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# THE SCHUR MULTIPLIER AND CENTRAL EMBEDDING PROBLEMS

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Abstract: The purpose of this note is to complete previous observations on connections between central embedding problems for a profinite group and its Schur multiplier. In addition the concept of uniform triviality of the Schur multiplier is introduced.

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## § 1. Introduction

In this note we provide conditions for the triviality of the Schur multiplier of a profinite group in terms of central embedding or lifting problems for this group, thereby completing and improving upon certain results in $[MO]$, $[NO]$, $[O1]$, $[O2]$, $[O3]$. Lifting problems were already used to characterize profinite $p$-groups of cohomological dimension 1; comp. e.g. $[SE2]$, Chapter I, 3.4.

Let $\mathcal{G}$ be a profinite group and let $A$ denote a finite abelian group. An *embedding or lifting problem* $E(G, A, c)$ *for* $\mathcal{G}$ *with kernel* $A$ consists of a finite quotient group $G$ of $\mathcal{G}$ such that $A$ is a $G$-module and a 2-cocycle $c$ on $G$ with values in $A$, i.e. $c : G \times G \to A$ is a map which satisfies $c(s,t)c(st,r) = c(s,tr)s(c(t,r))$ for all $s,t,r \in G$. A *solution* for this embedding problem is a homomorphism $\phi : \mathcal{G} \to G(c)$, where $G(c)$ denotes the group extension of $G$ with kernel $A$ which is defined by $c$, such that the composition of $\phi$ with the natural epimorphism $G(c) \to G$ is equal to the given epimorphism $\mathcal{G} \to G$.

For any profinite group $\mathcal{V}$, any discrete $\mathcal{V}$-module $M$ and any integer $q \geq 0$ denote by $H^q(\mathcal{V}, M)$ the $q$-th cohomology group of $\mathcal{V}$ with respect to $M$.

According to $[H]$, 1.1, the following result holds.

(1.1) **Proposition** *The embedding problem $E(G, A, c)$ for $\mathcal{G}$ is solvable if and only if the cohomology class $(c) \in H^2(G, A)$ belongs to the kernel of the inflation homomorphism*

$$\inf_G^{\mathcal{G}} = \inf_G^{\mathcal{G}}(A) : H^2(G, A) \to H^2(\mathcal{G}, A)$$

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Two embedding problems $E(G_1, A, c_1)$, $E(G_2, A, c_2)$ for $\mathcal{G}$ with the same kernel $A$ are said to be *equivalent* if $\inf_{G_1}^{\mathcal{G}}((c_1)) = \inf_{G_2}^{\mathcal{G}}((c_2))$. This is an equivalence relation on the set of all embedding problems for $\mathcal{G}$ with kernel $A$. The embedding problem $E(G, A, c)$ for $\mathcal{G}$ is said to be *cyclic* if $G$ is cyclic, and it is said to be *central* if the action of $G$ on $A$ is trivial. A central embedding problem for $\mathcal{G}$ is said to have a *cyclic reduction,* see[$O1$], if it is equivalent to a cyclic central embedding problem for $\mathcal{G}$.

For $\mathcal{G}$ and every finite quotient group $G$ of $\mathcal{G}$ the group $C$ of all roots of unity in $\mathbb{C}$ and all the finite cyclic subgroups $C_m$ of order $m \in \mathbb{N}$ are regarded as $\mathcal{G}$-resp. $G$-modules with respect to the trivial action of $\mathcal{G}$ resp. $G$.

We recall, see [$O2$], §2, p. 226, or [$MO$], section 1, that a central embedding problem $E(G, C_m, c)$ for $\mathcal{G}$ is said to be *weakly solvable* if there is some multiple $m'$ of $m$ such that the central embedding problem $E(G, C_{m'}, c_{m,m'})$ for $\mathcal{G}$, where

$$c_{m,m'} : G \times G \xrightarrow{c} C_m \hookrightarrow C_{m'}$$

denotes the 2-cocycle which is obtained by composing $c$ with the natural embedding $C_m \hookrightarrow C_{m'}$, is solvable.

## § 2. Cyclic reduction and weak solvability of central embedding problems

Let $\mathcal{G}$ be a profinite group which acts trivially on $C$. Then $H^2(\mathcal{G}, C)$ is called the *Schur multiplier of* $\mathcal{G}$.

(2.1) **Proposition** *The following statements are equivalent.*
*(a)* $H^2(\mathcal{G}, C) = 1$
*(b) Every central embedding problem* $E(G, C_m, c)$ *for* $\mathcal{G}$ *is weakly solvable*
*(c) Every central embedding problem* $E(G, C_m, c)$ for $\mathcal{G}$ *has a cyclic reduction, i.e. is equivalent to a cyclic central embedding problem for* $\mathcal{G}$

The equivalence of statements (a) and (b) is implicit in [$O2$], and it is shown in the proof of proposition (1) of [$O1$] that statement (a) implies statement (c). For the sake of completeness we give a full proof for this proposition: Assume that statement (a) holds. Let $E(G, C_m, c)$ be a central embedding problem for $\mathcal{G}$. Then there is a continuous function $\alpha : \mathcal{G} \to C$ such that $\inf_G^{\mathcal{G}}(c) = \delta\alpha$. Put $\mathcal{N} := \{s \in \mathcal{G} : \alpha(s) \in C_m\}$. $\mathcal{N}$ is the kernel of the continuous homomorphism

$$\overline{\alpha} : \mathcal{G} \to C/C_m, \quad s \mapsto \alpha(s) \bmod C_m.$$

Hence $\mathcal{N} \le \mathcal{G}$ is an open normal subgroup, and the quotient group $Z := \mathcal{G}/\mathcal{N}$ is a finite cyclic group. The central embedding problem $E(Z, C_m, \delta\alpha)$ for $\mathcal{G}$ is equivalent to $E(G, C_m, c)$ because $\inf_G^{\mathcal{G}}((c)) = \inf_Z^{\mathcal{G}}((\delta\alpha))$. Hence statement (c) holds. It is also easily seen that statement (a) implies statement (b). In fact, the equation $\inf_G^{\mathcal{G}}(c) = \delta\alpha$ shows that $\alpha^m : \mathcal{G} \to C$ is a continuous homomorphism, hence there is a positive integer $n$ such that $\alpha^{mn} = 1$. This in conjunction

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with (1.1) shows that the central embedding problem $E(G, C_{mn}, c_{m,mn})$ for $\mathcal{G}$ is solvable and therefore statement (b) holds. Assume that statement (b) holds. Take $(t) \in H^2(\mathcal{G}, C)$. Then there is a finite quotient group $G$ of $\mathcal{G}$ and some $(c) \in H^2(G, C_m)$, where $m =$ order of $(t)$, such that $(t) = (\iota_{C_m,C} \circ \inf_G^{\mathcal{G}})((c))$, where $\iota_{C_m,C} : H^2(\mathcal{G}, C_m) \to H^2(\mathcal{G}, C)$ is the homomorphism induced by the natural embedding $C_m \hookrightarrow C$. Let $m'$ be a multiple of $m$ such that the central embedding problem $E(G, C_{m'}, c_{m,m'})$ for $\mathcal{G}$ is solvable. Then according to (1.1) there is a continuous function $\alpha : \mathcal{G} \to C_{m'}$ such that $(\delta\alpha) = \inf_G^{\mathcal{G}}((c_{m,m'})) = (\inf_G^{\mathcal{G}} \circ \iota_{m,m'})((c)) = (\iota_{m,m'} \circ \inf_G^{\mathcal{G}})((c))$, where $\iota_{m,m'} : H^2(-, C_m) \to H^2(-, C_{m'})$ is the respective homomorphism induced by the natural embedding $C_m \hookrightarrow C_{m'}$. Hence $(t) = (\iota_{C_{m'},C} \circ \iota_{m,m'} \circ \inf_G^{\mathcal{G}})((c)) = (\delta(\iota_{C_{m'},C}(\alpha)))$. Therefore statement (a) holds. Assume that statement (c) holds. In order to show that statement (b) holds it is sufficient to show that every cyclic central embedding problem $E(Z, C_m, c)$ for $\mathcal{G}$ is weakly solvable. Since $H^2(Z, C)$ is trivial there is a function $\alpha : Z \to C$ such that $\iota_{C_m,C}((c)) = (\delta\alpha)$. Hence $\alpha^m : Z \to C$ is a homorphism and therefore $\alpha^{m|Z|} = 1$. This implies that the central embedding problem $E(Z, C_{m|Z|}, c_{m,m|Z|})$ is solvable and therefore $E(Z, C_m, c)$ is weakly solvable.

Assume that the central embedding problem $E(G, C_m, c)$ for $\mathcal{G}$ is weakly solvable. The *lifting index* $l = l(E(G, C_m, c))$ of $E(G, C_m, c)$ is defined as the smallest multiple $m'$ of $m$ such that the central embedding problem $E(G, C_{m'}, c_{m,m'})$ for $\mathcal{G}$ is solvable. Let $E(Z, C_m, d)$ be a cyclic reduction of $E(G, C_m, c)$. Since the homomorphisms $\iota_{m,m'} : H^2(-, C_m) \to H^2(-, C_{m'})$ commute with the corresponding inflation homomorphisms we have

(2.2) **Remark** $l(E(G, C_m, c)) = l(E(Z, C_m, d))$

Assume that $H^2(\mathcal{G}, C) = 1$. Then the *reduction index* $r = r(E(G, C_m, c))$ of a central embedding problem $E(G, C_m, c)$ for $\mathcal{G}$ is defined to be the minimal order of a cyclic quotient group $Z$ of $\mathcal{G}$ such that $E(G, C_m, c)$ is equivalent to a central embedding problem $E(Z, C_m, d)$ for $\mathcal{G}$.

(2.3) **Proposition** Assume that $H^2(\mathcal{G}, C) = 1$. Then $l(E(G, C_m, c)) = m \cdot r(E(G, C_m, c))$

Proof: Denote by $E(Z, C_m, d)$ a cyclic reduction of $E(G, C_m, c)$ such that $l(E(G, C_m, c)) = l(E(Z, C_m, d))$. Since $Z$ is cyclic there is a function $\alpha : Z \to C$ such that $\iota_{C_m,C}((d)) = (\delta(\alpha))$. Hence $\alpha^m \in Hom(Z, C)$ and therefore $\alpha^{mr} = 1$ where $r = \exp(Z)$. Hence $\iota_{m,mr}((d)) = (1)$, and therefore $l(G, C_m, c)$ divides $mr(E(G, C_m, c))$. On the other hand for $l = l(E(G, C_m, c))$ there is a function $\alpha : \mathcal{G} \to C_l$ such that $\inf_G^{\mathcal{G}}(\iota_{m,l}(c)) = \delta\alpha$ and $\alpha^m \in Hom(\mathcal{G}, C_l)$ is of order $l/m$. Put $Z := \mathcal{G}/Ker(\alpha^m)$. The central embedding problem $E(Z, C_m, \delta\alpha)$ for $\mathcal{G}$ is a cyclic reduction of $E(G, C_m, c)$. Hence $r$ divides $l/m$. The assertion follows.

Let $k$ be a field with separable algebraic closure $\overline{k}$ and absolute Galois group $G_k = G(\overline{k}/k)$.

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(2.4) **Examples** We list some examples of fields $k$ such that its absolute Galois group $\mathcal{G} = G_k$ has trivial Schur multiplier.

(a) $k$ a local number field; see [$SE2$], Chapter II, 5.3, Proposition 15.

(b) $k$ a global number field; see [$T$]; [$SE1$], §6.

(c) $k$ is an extension of transcendence degree 1 over an algebraically closed field; see [$SE2$], Chapter II, 3.3, b).

(d) $k$ is the rational function field in one variable over a real closed field; see [$LO$], 5, (ii).

In certain situations it is possible to bound the lifting index of a central embedding problem $E(G, C_m, c)$ for $G_k$ by using central simple algebras, see [$O2$], (4.5). This method is based on [$H$], 3.8, and yields the following result.

(2.5) **Proposition** *Let $E(G, C_m, c)$ be a central embedding problem for $G_k$ with $G = G(K/k)$. Let $m'$ be a mutiple of $m$ and denote by $\mu_{m'} \leq \overline{k}^*$ the group of roots of unity of order $m'$. Assume that the cyclotomic extension $k(\mu_{m'})/k$ is cyclic and that the central simple crossed product algebra $\Gamma(K(\mu_{m'})/k(\mu_{m'}), c)$ over $k(\mu_{m'})$ splits. The the lifting index $l = l(E(G, C_m, c))$ divides $m'$.*

For other relations between the Schur multiplier of $G_k$ and the Brauer group of $k$ see [$LO$].

## § 3. Uniform triviality of the Schur multiplier of a profinite group

Let $\mathcal{G}$ be a profinite group.

**Definition** The Schur multiplier $H^2(\mathcal{G}, C)$ of $\mathcal{G}$ is said to be *uniformly trivial* if it is trivial and if there is a positive integer $r$ depending only on $\mathcal{G}$ such that the reduction index of every central embedding problem of the form $E(G, C_m, c)$ for $\mathcal{G}$ divides $r$. The smallest positive integer with this property is denoted by $r(\mathcal{G})$, and if $r(\mathcal{G})$ exists then we indicate this fact sometimes by writing $H^2(\mathcal{G}, C) \equiv 1$ or by $H^2(\mathcal{G}, C) \equiv_{r(\mathcal{G})} 1$.

**Examples** (a) Denote by $k$ a local number field. Then for every prime number $p$ the maximal pro-$p$-quotient group $G_k(p)$ of the absolute Galois group $G_k$ satisfies $H^2(G_k(p), C) \equiv_{r(G_k(p))} 1$, where $r(G_k(p))$ depends only on $G_k(p)$; this follows by applying (2.3) to the results in [$MOM$], §2 and §3, which lead to upper bounds for the lifting index in terms of the roots of unity of $p$-power order contained in $k$, and from local class field theory.

(b) Let $k$ be a number field, let $p$ be a prime number and let $S$ denote a finite set of places of $k$ which contains all places above $p$ and infinity. Then the Galois group $G_k(S, p)$ of the maximal $p$-extension of $k$ which is unramified outside $S$ satisfies $H^2(G_k(S, p), C) \equiv_{r(G_k(S,p)} 1$ where $r(G_k(S, p))$ depends only on $G_k(S, p)$; in fact, a bound can be given by using the exponent of the $p$-torsion part of the maximal abelian profinite quotient group of $G_k(S, p)$; see [$MH$] in connection with [$M$], Theorem 3, p. 75, and [$NO$], (3.1), p. 12 ff.

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